Nov. 2nd, 2021

Quiz & Optimization Problems

Suppose you have one rabbit.

Now suppose someone gives you one more rabbit.

Now, if you count your rabbits, you have two rabbits. So one rabbit plus one rabbit equals two rabbits. So one plus one equals two.

$$1 + 1 = 2$$

And that is how arithmetic is done.

Now that you understand the basic idea behind arithmetic, let's take a look at a simple easy-to-understand example that puts into practice what we just learned.

Try It Out
Example 1.7

$$\log \Pi(N) = \left(N + \frac{1}{2}\right)\log N - N + A - \int_N^\infty \frac{\overline{B}_1(x)\mathrm{d}x}{x}, \quad A = 1 + \int_1^\infty \frac{\overline{B}_1(x)\mathrm{d}x}{x}$$

$$\log \Pi(s) = \left(s + \frac{1}{2}\right)\log s - s + A - \int_0^\infty \frac{\overline{B}_1(t)\mathrm{d}t}{t + s}$$

$$\begin{aligned}
\log \Pi(s) &= \lim_{n\to\infty}\left[s\log(N+1) + \sum_{n=1}^{N}\log n - \sum_{n=1}^{N}\log(s+n)\right] \\
&= \lim_{n\to\infty}\left[s\log(N+1) + \int_1^N \log x\,\mathrm{d}x - \frac{1}{2}\log N + \int_1^N \frac{\overline{B}_1(x)\mathrm{d}x}{x}\right. \\
&\quad - \int_1^N \log(s+x)\,\mathrm{d}x - \frac{1}{2}[\log(s+1) + \log(s+N)] \\
&\quad \left. - \int_1^N \frac{\overline{B}_1(x)\mathrm{d}x}{s+x}\right] \\
&= \lim_{n\to\infty}\left[s\log(N+1) + N\log N - N + 1 + \frac{1}{2}\log N + \int_1^N \frac{\overline{B}_1(x)\mathrm{d}x}{x}\right. \\
&\quad - (s+N)\log(s+N) + (s+N) + (s+1)\log(s+1) \\
&\quad \left. - (s+1) - \frac{1}{2}\log(s+1) - \frac{1}{2}\log(s+N) - \int_1^N \frac{\overline{B}_1(x)\mathrm{d}x}{s+x}\right] \\
&= \left(s + \frac{1}{2}\right)\log(s+1) + \int_1^\infty \frac{\overline{B}_1(x)\mathrm{d}x}{x} - \int_1^\infty \frac{\overline{B}_1(x)\mathrm{d}x}{s+x} \\
&\quad + \lim_{n\to\infty}\left[s\log(N+1) + \left(N + \frac{1}{2}\right)\log N\right.
\end{aligned}$$

Every STEM class...

Quiz 5.

problem 1. Sketch the graph of the function $f(x) = x^2 + \frac{1}{x} - 1$.

domain $(-\infty, 0) \cup (0, +\infty)$.

Warning: 0 is NOT an inflection pt b/c 0 is not in the domain of f.

asymptote: $\lim_{x \to \pm\infty} f(x) = +\infty$. so NO horizontal asymptotes.

$\lim_{x \to 0+} f(x) = +\infty$, and $\lim_{x \to 0-} f(x) = -\infty$, $x=0$ is a vertical asymptote.

$f'(x) = 2x - \frac{1}{x^2}$, $f'(x) = 0 \rightsquigarrow 2x - \frac{1}{x^2} = 0 \rightsquigarrow x = \sqrt[3]{\frac{1}{2}}$ ← crit pt with value.

$f(\sqrt[3]{\frac{1}{2}}) = \sqrt[3]{\frac{1}{4}} + \frac{1}{\sqrt[3]{\frac{1}{2}}} - 1 = \frac{1}{\sqrt[3]{4}} + \sqrt[3]{2} - 1$

$$\begin{cases} f'(x) < 0, & x < 0 \\ f'(x) < 0, & 0 < x < \sqrt[3]{\frac{1}{2}} \\ f'(x) > 0, & x > \sqrt[3]{\frac{1}{2}}. \end{cases}$$

$x = \sqrt[3]{\frac{1}{2}}$ is a local minimum.

$f''(x) = 2 + \frac{2}{x^3} = 0 \rightsquigarrow x = -1$ ← inflection pt.

$$\begin{cases} x < -1: & f''(x) > 0 \\ -1 < x < 0: & f''(x) < 0 \\ x > 0: & f''(x) > 0. \end{cases}$$

concave up in $(-\infty, -1)$

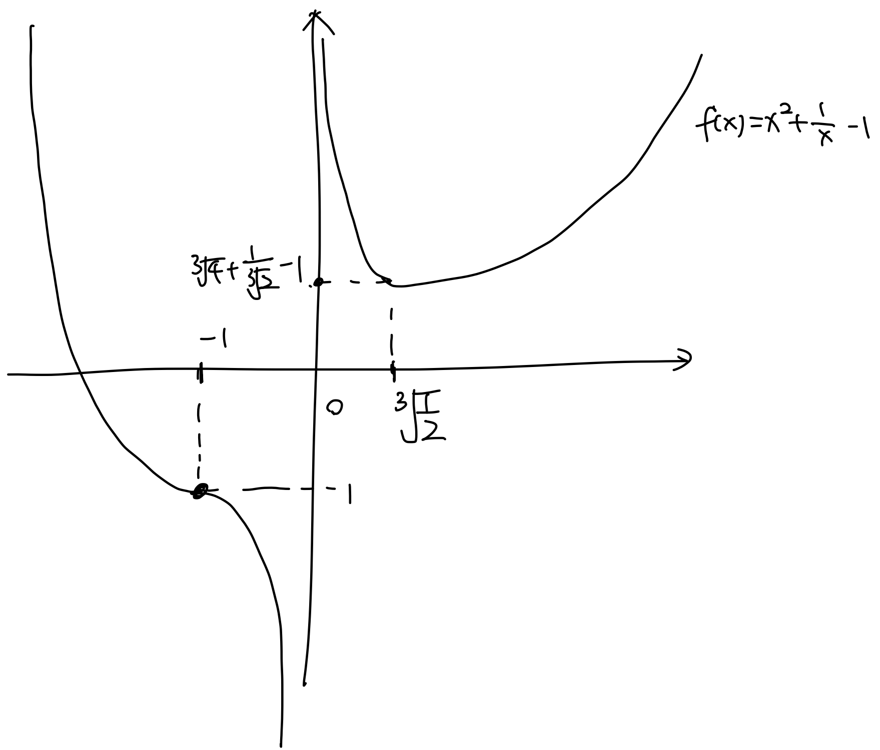
down in $(-1, 0)$

up in $(0, +\infty)$

$f(-1) = -1$.

Quiz 5.

problem 1. Sketch the graph of the function $f(x) = x^2 + \frac{1}{x} - 1$.



Problem 2. (8 Points) Let R be a rectangle with length a and side b, so that $\frac{a}{b} = \frac{4}{3}$. Let d be the length of the diagonal. If d increases at the rate of 2cm/s, what's the rate that the area A of R grows when $d = 5$cm?

$D^2 = a^2 + b^2$. $A = ab$.

$\frac{a}{b} = \frac{4}{3} \rightsquigarrow a = \frac{4}{3}b$. (1)

$\frac{dD}{dt} = 2 \text{ cm/s}$

$\frac{dA}{dt} = a\frac{db}{dt} + \frac{da}{dt}b$.

$\frac{da}{dt} = \frac{4}{3}\frac{db}{dt}$. (2)

$D^2 = a^2 + b^2 = \left(\frac{4}{3}b\right)^2 + b^2 = \frac{16b^2}{9} + b^2 = \frac{25}{9}b^2$, so $D = \frac{5}{3}b$ (3).

$2D\frac{dD}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$.

$\frac{dD}{dt} = \frac{5}{3}\frac{db}{dt}$ (4).

(4) reads $\frac{db}{dt} \cdot \frac{5}{3} = 2$ $\frac{db}{dt} = 2 \times \frac{3}{5} = \frac{6}{5} = 1.2$ cm/s. Plugging in (2): $\frac{da}{dt} = \frac{4}{3} \times 1.2$ cm/s $= 1.6$ cm/s.

$D = 5$cm plugging in (3): $b = \frac{3}{5} \cdot D = \frac{3}{5} \times 5$ cm $= 3$ cm. plugging in (1): $a = \frac{4}{3} \times 3$ cm $= 4$ cm.

Finally: $\frac{dA}{dt} = 4 \times 1.2 + 1.6 \times 3 = 4.8 + 4.8 = 9.6$ cm²/s.

Problem 2. (8 Points) Let R be a rectangle with length a and side b, so that $\frac{a}{b} = \frac{4}{3}$. Let d be the length of the diagonal. If d increases at the rate of 2cm/s, what's the rate that the area A of R grows when $d = 5$cm?

$$D^2 = a^2 + b^2.$$

$$2D\frac{dD}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}.$$

$$D\frac{dD}{dt} = a\frac{da}{dt} + b\frac{db}{dt}.$$

$D = 5$, $\frac{dD}{dt} = 2$

$$10 = a\frac{da}{dt} + b\frac{db}{dt}. \quad (5).$$

$a = 4$ cm, $b = 3$ cm

$$10 = 4\frac{da}{dt} + 3\frac{db}{dt}.$$

$$= 4\cdot\left(\frac{4}{3}\frac{db}{dt}\right) + 3\frac{db}{dt}$$

$$= \frac{16+9}{3}\frac{db}{dt} = \frac{25}{3}\frac{db}{dt}.$$

$$\frac{db}{dt} = \frac{3}{25}\times 10 = \frac{6}{5} \text{ cm/s}.$$

$$\frac{da}{dt} = \frac{4}{3}\frac{db}{dt} = \frac{4}{3}\times\frac{6}{5} = \frac{8}{5} \text{ cm/s}.$$

get $\frac{dA}{dt}$.

Optimization Problem.

43. (a) If $C(x)$ is the cost of producing x units of a commodity, then the **average cost** per unit is $c(x) = C(x)/x$. Show that if the average cost is a minimum, then the marginal cost equals the average cost. $C'(x) = c(x)$.

(b) If $C(x) = 16{,}000 + 200x + 4x^{3/2}$, in dollars, find (i) the cost, average cost, and marginal cost at a production level of 1000 units; (ii) the production level that will minimize the average cost; and (iii) the minimum average cost.

→ formula for $C'(x)$ and $c'(x)$.

→ find this value.

$$c'(x) = \left(\frac{C(x)}{x}\right)' = \frac{xC'(x) - C(x)}{x^2} = 0$$

$$\rightsquigarrow xC'(x) - C(x) = 0$$

$$\rightsquigarrow C'(x) = \frac{C(x)}{x} = c(x).$$

$$C'(x) = 200 + 6\sqrt{x}.$$

$$c(x) = \frac{C(x)}{x} = \frac{16000}{x} + 200 + 4\sqrt{x}.$$

$$c'(x) = -\frac{16000}{x^2} + \frac{2}{\sqrt{x}}$$

$$\sqrt{1000} = \sqrt{100 \times 10} = 10\sqrt{10}.$$

$$(2\sqrt{x})' = (2x^{\frac{1}{2}})' = \frac{1}{2} \cdot 2 \cdot x^{\frac{1}{2}-1} = x^{-\frac{1}{2}}.$$

increasing

$$c'(x) = -\frac{16000}{x^2} + \frac{2}{\sqrt{x}} = \frac{2x^{\frac{3}{2}} - 16000}{x^2} = 0$$

$$2x^{\frac{3}{2}} - 16000 = 0 \quad 2x^{\frac{3}{2}} = 16000 \quad x^{\frac{3}{2}} = 8000.$$

solution for $x_0 = \sqrt[3]{(8000)^2}$

$$= \left(\sqrt[3]{8000}\right)^2 = 20^2 = 400$$

$x < 400 : c'(x) < 0 \rightsquigarrow x_0 = 400$ is a (local) minimum

$x > 400 : c'(x) > 0.$

Anti-derivative.

$f(x)$ $\quad F'(x)=f$, then F is the **anti-derivative** of f.

Find f: 1) $f''(\theta)=\sin\theta+\cos\theta$, $f(0)=3$, $f'(0)=4$; 2) $f'(t)=1.5\sqrt{t}$, $f(4)=10$.

$f'(t)=\frac{3}{2}\sqrt{t}$,

$\left(t^{\frac{3}{2}}\right)'=\frac{3}{2}t^{\frac{1}{2}}=\frac{3}{2}\sqrt{t}$.

$\sqrt{4^3}=2^3=8$.

$f(t)=t^{\frac{3}{2}}\boxed{+C.}$

$f(4)=4^{\frac{3}{2}}+C=10$

$8+C=10$

$C=2$

so $f(t)=t^{\frac{3}{2}}+2$.

Reminder:

- Quiz 6: Optimization (16 pts) + Antiderivative (8 pts). ← 2 prbs?

  practice problems: problem 27, 31, 41 and 42 in Chpt 3 Review

- OH today 1-2 pm, appointments accepted.

See U On Thursday!